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A theorem on the local isometric embedding of empty space–times in a space of constant curvature

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Abstract. It is shown that no non-flat empty space–time can be embedded locally and isometrically in a five-dimensional space of nonzero constant curvature.

1. Introduction

The theorem that no non-flat n dimensional Riemannian space with zero Ricci tensor can be embedded, locally and isometrically, in an $n+1$ dimensional pseudo-euclidean (flat) space is often quoted in the literature (Eisenhart 1925 p. 200). The usual proof of this theorem does not hold for indefinite metrics. This fact has been pointed out by Szekeres (1966) who gives a proof valid for the (empty) space–times of general relativity. In this paper an extension of Szekeres' result is proved, namely: *Theorem.* No non-flat empty space–time can be embedded locally and isometrically in a five-dimensional space of nonzero constant curvature.

An n dimensional Riemannian space with metric g_{ij} can be embedded locally and isometrically in an $n+1$ dimensional space of constant curvature K_0 if and only if there exists a symmetric tensor a_{ij} satisfying the following equations (Eisenhart 1925 p. 210):

Gauss equation

$$R_{ijkl} = e(a_{ik}a_{jl} - a_{il}a_{jk}) + K_0(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{1.1}$$

Codazzi equation

$$a_{ij;k} - a_{ik;j} = 0. \tag{1.2}$$

Here $e = \pm 1$ and R_{ijkl} is the curvature tensor of the n dimensional Riemannian space.

2. Proof of theorem

Contracting equation (1.1) yields, on putting $R_{jk} = 0$,

$$\epsilon(a_{ik}a_j{}^i - a_i{}^i a_{jk}) = 3K_0 g_{jk}.$$

This equation, with $3K_0$ replaced by Λ, has been investigated by Szekeres (1966 equation 14) in a different context. Szekeres proves that the tensor a_{jk} must take one of the two canonical forms

$$(a) \quad a_{jk} = \lambda g_{jk} \qquad -\epsilon K_0 = \lambda^2$$

$$(b) \quad a_{jk} = \lambda g_{jk} + 2\lambda u_j u_k - 2\lambda s_j s_k \qquad 3\epsilon K_0 = -\lambda^2$$

where u_j, s_j are vectors satisfying $-u_j u^j = s_j s^j = 1$, $u_j s^j = 0$.

Substituting the canonical form (a) into equation (1.1) gives $R_{ijkl} = 0$ which contradicts the hypothesis of the theorem.

The canonical form (b) can be rewritten as

$$a_{jk} = |3K_0|^{1/2} g_{jk} - |12K_0|^{1/2}(l_j n_k + n_k l_j) \tag{2.1}$$

where the null vectors l_j, n_j are defined by

$$l_j = (s_j + u_j)/\sqrt{2} \qquad n_j = (s_j - u_j)/\sqrt{2}.$$

Substituting (2.1) into (1.1) yields

$$\begin{aligned} R_{ijkl} = {} & 4K_0(g_{ik}g_{jl} - g_{il}g_{jk}) - 6K_0\{g_{ik}(l_j n_l + l_l n_j) - g_{il}(l_j n_k + l_k n_j)\} \\ & + 12K_0\{(n_k l_l - n_l l_k)(l_i n_j - n_i l_j)\}. \end{aligned} \tag{2.2}$$

Substituting (2.1) into the Codazzi equation (1.2) yields

$$l_{i;k}n_j + l_{j;k}n_i + l_i n_{j;k} + l_j n_{i;k} - l_{i;j}n_k - l_{k;j}n_i - l_i n_{k;j} - l_k n_{i;j} = 0. \tag{2.3}$$

Contracting this equation with l^j yields

$$l_{i;k} + l_i n_{j;k} l^j - l_{i;j} l^j n_k - l_{k;j} l^j n_i - l_i n_{k;j} l^j - l_k n_{i;j} l^j = 0. \tag{2.4}$$

Contracting equation (2.4) with l^i and n^i gives

$$l_{k;j}l^j = l_k n^i l_{i;j} l^j \qquad \text{and} \qquad n_{k;j}l^j = -n_k n^i l_{i;j} l^j.$$

Using these two results equation (2.4) becomes

$$l_{i;k} = -l_i A_k \tag{2.5}$$

where

$$A_k = n_{j;k} l^j.$$

By a similar argument

$$n_{i;k} = n_i A_k \tag{2.6}$$

and then the Codazzi equation (2.3) is identically satisfied by virtue of equations (2.5) and (2.6).

Differentiating equation (2.5) yields

$$l_{i;kj} = -l_{i;j}A_k - l_i A_{k;j} = l_i A_j A_k - l_i A_{k;j}. \tag{2.7}$$

Hence

$$l_{i;[kj]} = 2R^s{}_{ikj} l_s = -l_i A_{[k;j]}.$$

From this equation it follows that

$$l_{[m}R_{i]skj} l^s = 0.$$

Substituting the expression (2.2) into this equation gives

$$-4K_0 l_{[m} g_{i][k} l_{j]} = 0.$$

Hence for an embedding K_0 must vanish and this proves the theorem.

References

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